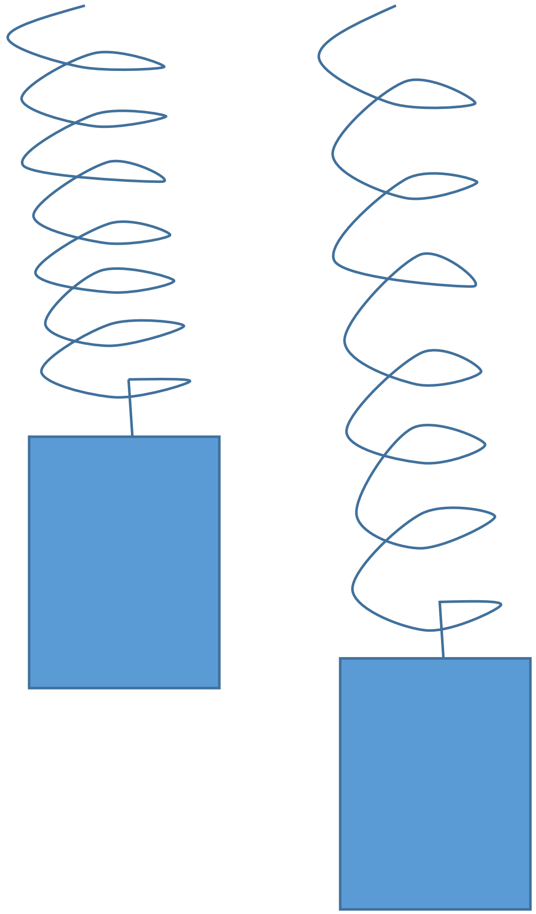
12. Wilberforce Pendulum

Sergej Faletič
University of Ljubljana, Faculty of Mathematics and Physics
Slovenia

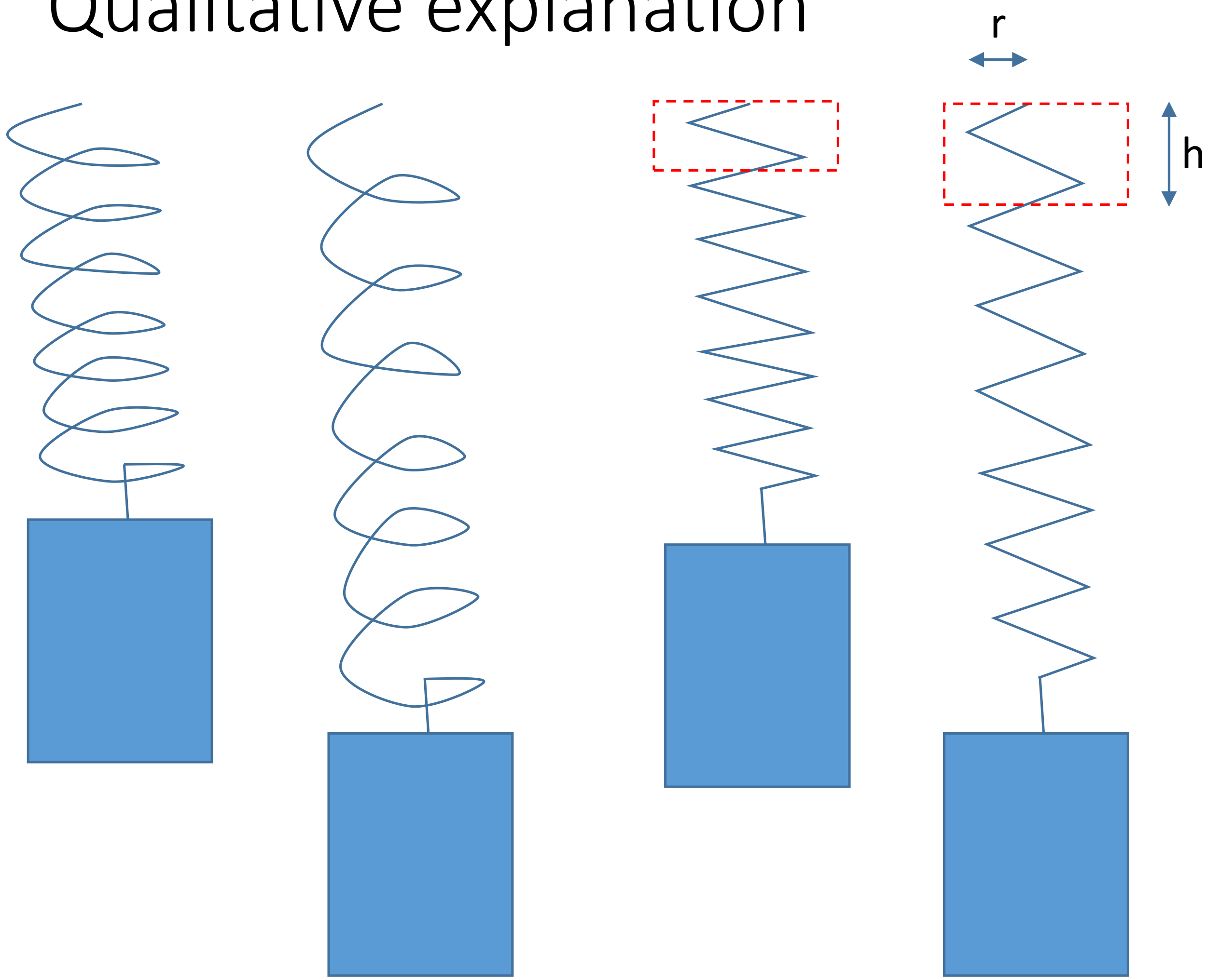
The phenomenon

- https://www.youtube.com/watch?v=S42ILTlnfZc

Qualitative explanation



Qualitative explanation

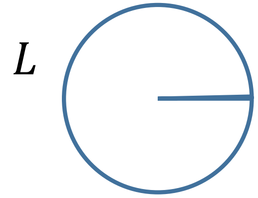


Length:

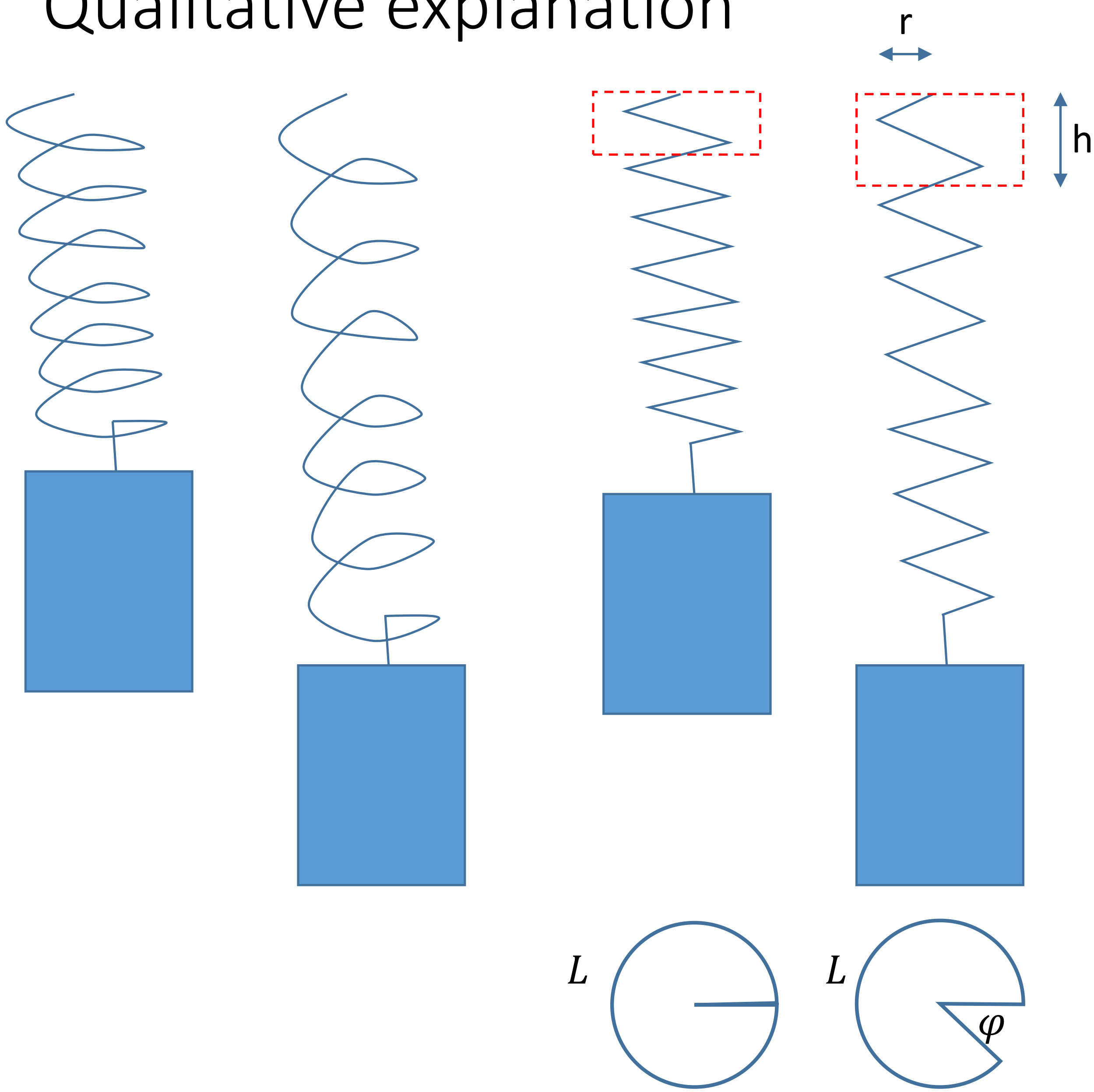
$$L = \sqrt{h^2 + (2\pi r)^2}$$

radius might change, but let us ignore this for now

$r = r(h)$



Qualitative explanation



Length:

$$L = \sqrt{h^2 + (2\pi r)^2}$$

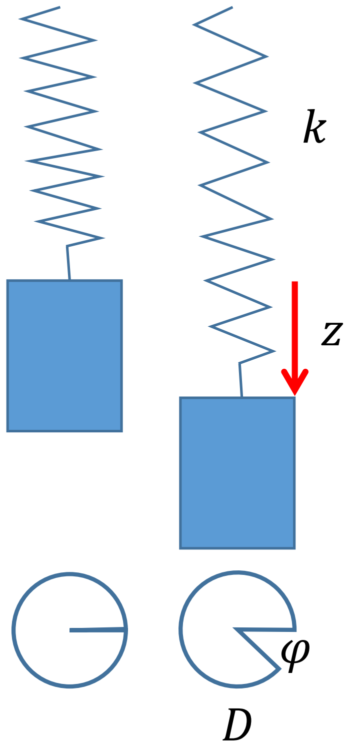
radius might change, but let us ignore this for now

$r = r(h)$

Calculation

$$m\frac{d^2z}{dt^2} = -kz$$

$$I\frac{d^2\varphi}{dt^2} = -D\varphi$$



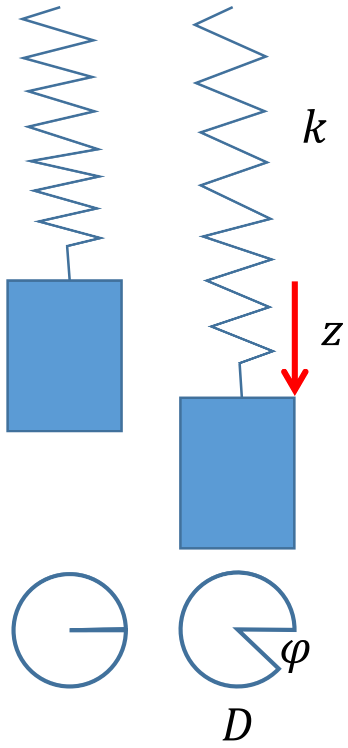
Calculation

$$m\frac{d^2z}{dt^2} = -kz$$

$$I\frac{d^2\varphi}{dt^2} = -D\varphi$$

$$z(t) = A\cos(\omega_z t + \delta) \qquad \omega_z = \sqrt{k/m}$$

$$\varphi(t) = B\cos(\omega_\varphi t + \delta) \qquad \omega_\varphi = \sqrt{D/I}$$



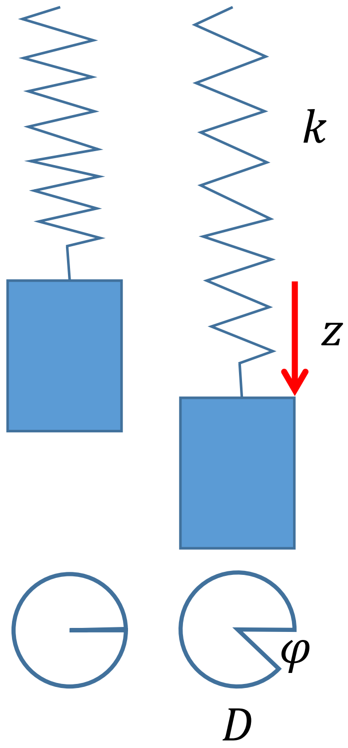
Calculation

$$m\frac{d^2z}{dt^2} = -kz - \xi\varphi$$

$$I\frac{d^2\varphi}{dt^2} = -D\varphi - \xi z$$

$$z(t) = A\cos(\omega_z t + \delta) \qquad \omega_z = \sqrt{k/m}$$

$$\varphi(t) = B\cos(\omega_\varphi t + \delta) \qquad \omega_\varphi = \sqrt{D/I}$$



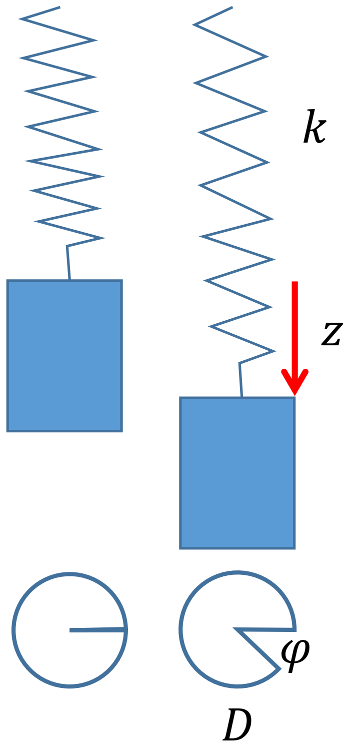
Calculation

$$m\frac{d^2z}{dt^2} = -kz - \xi\varphi$$

$$I\frac{d^2\varphi}{dt^2} = -D\varphi - \xi z$$

$$z(t) = A\cos(\omega t + \delta) \qquad \omega_z = \sqrt{k/m}$$

$$\varphi(t) = B\cos(\omega t + \delta) \qquad \omega_\varphi = \sqrt{D/I}$$



Calculation

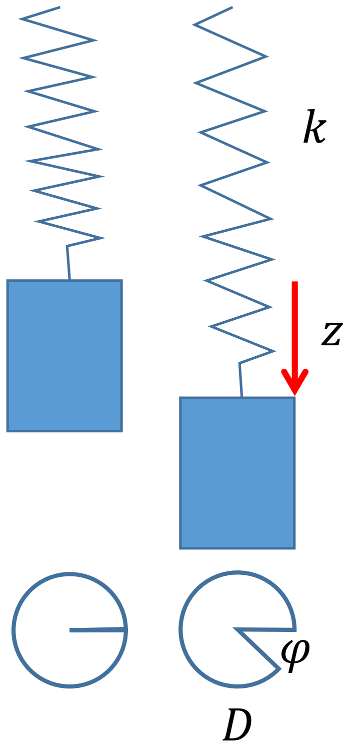
$$m\frac{d^2z}{dt^2} = -kz - \xi\varphi$$

$$I\frac{d^2\varphi}{dt^2} = -D\varphi - \xi z$$

$$z(t) = A\cos(\omega t + \delta) \qquad \omega_z = \sqrt{k/m}$$

$$\varphi(t) = B\cos(\omega t + \delta) \qquad \omega_\varphi = \sqrt{D/I}$$

$$\omega_\pm^2 = \frac{1}{2}\left[\omega_z^2 + \omega_\varphi^2 \pm \sqrt{\left(\omega_z^2 - \omega_\varphi^2\right)^2 + \frac{4\xi}{mI}}\right]$$



Calculation

$$m\frac{d^2z}{dt^2} = -kz - \xi\varphi$$

$$I\frac{d^2\varphi}{dt^2} = -D\varphi - \xi z$$

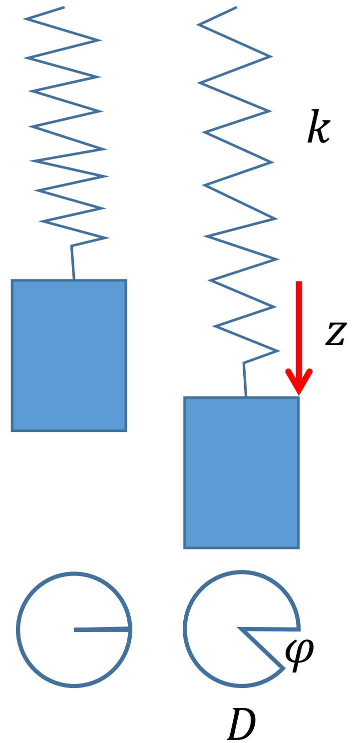
$$z(t) = A\cos(\omega t + \delta) \qquad \omega_z = \sqrt{k/m}$$

$$\varphi(t) = B\cos(\omega t + \delta) \qquad \omega_\varphi = \sqrt{D/I}$$

$$\omega_\pm^2 = \frac{1}{2}\left[\omega_z^2 + \omega_\varphi^2 \pm \sqrt{\left(\omega_z^2 - \omega_\varphi^2\right)^2 + \frac{4\xi}{mI}}\right]$$

$$z(t) = A_+\cos(\omega_+ t + \delta_+) + A_-\cos(\omega_- t + \delta_-),$$

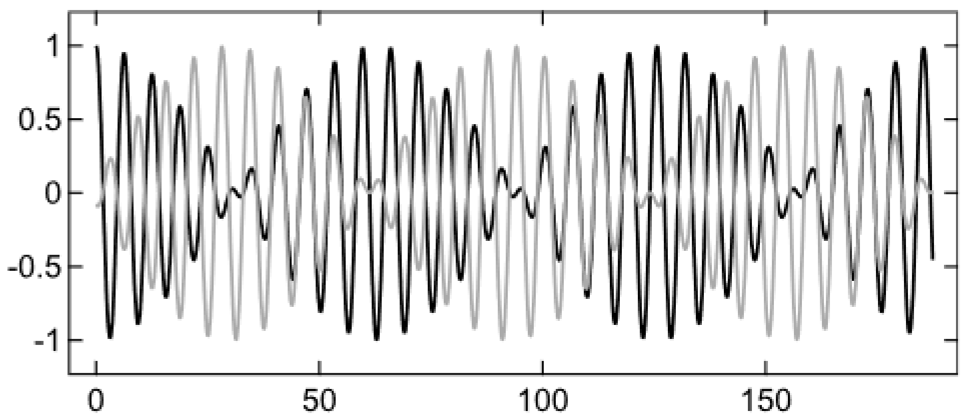
$$\varphi(t) = B_+\cos(\omega_+ t + \delta_+) + B_-\cos(\omega_- t + \delta_-).$$



Calculation results

$$z(t) = A_+ \cos(\omega_+ t + \delta_+) + A_- \cos(\omega_- t + \delta_-),$$

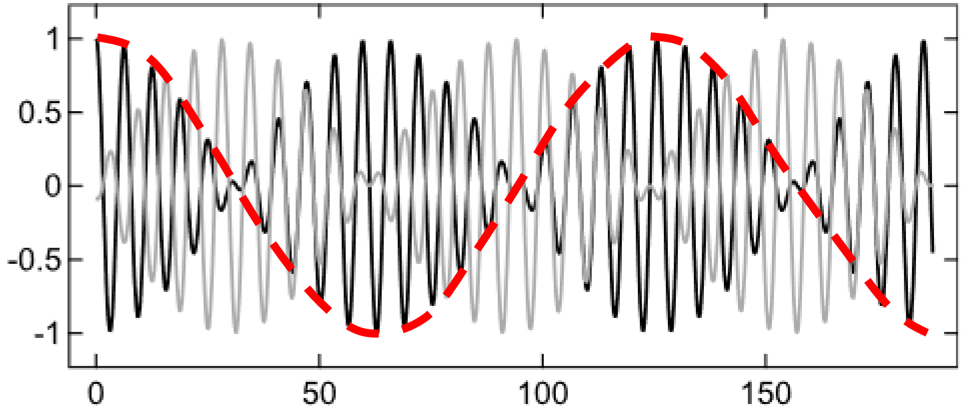
$$\varphi(t) = B_+ \cos(\omega_+ t + \delta_+) + B_- \cos(\omega_- t + \delta_-).$$



Calculation results

$$z(t) = A_+ \cos(\omega_+ t + \delta_+) + A_- \cos(\omega_- t + \delta_-),$$

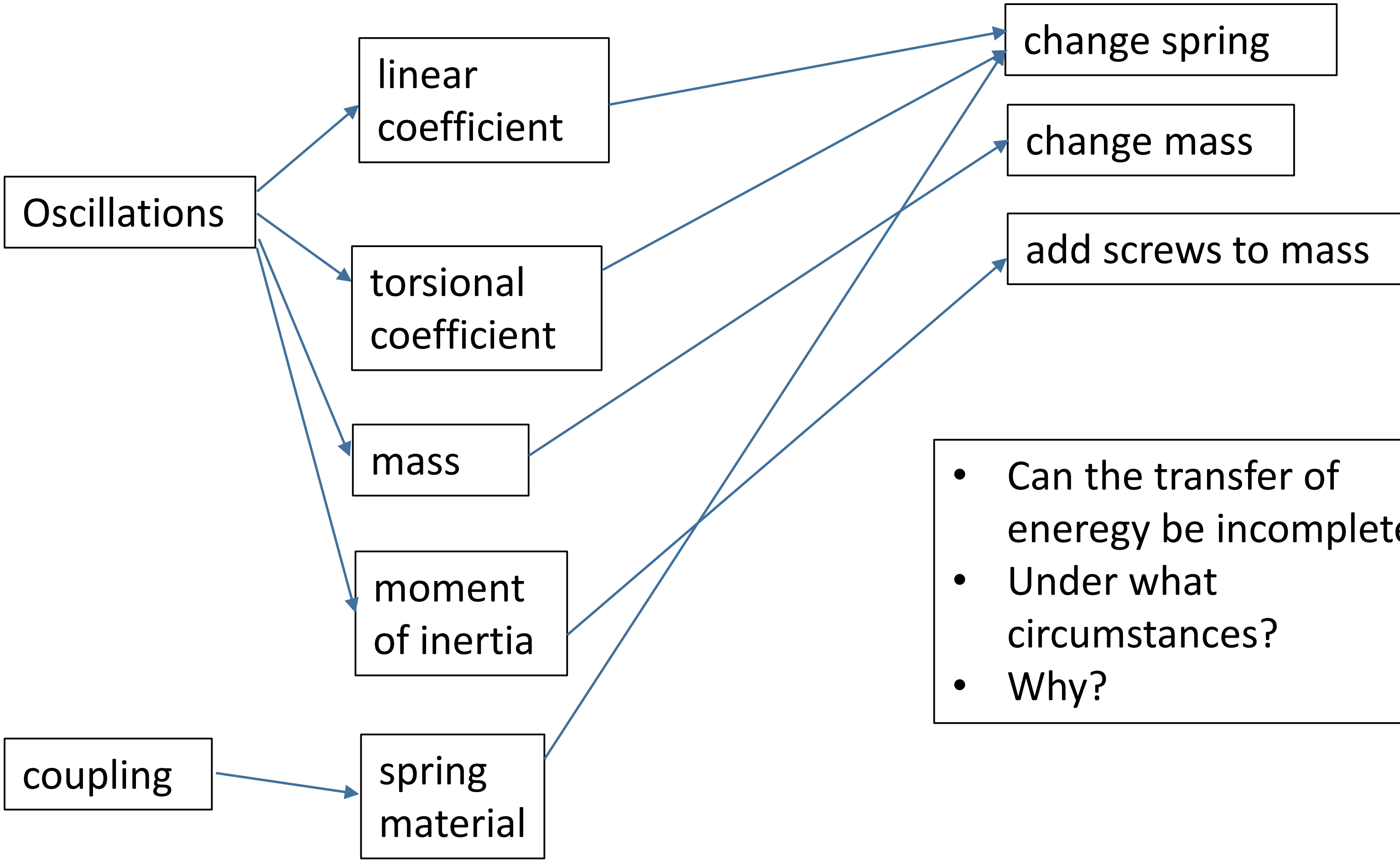
$$\varphi(t) = B_+ \cos(\omega_+ t + \delta_+) + B_- \cos(\omega_- t + \delta_-).$$



ω_b

$$\omega_\pm^2 = \frac{1}{2}\left[\omega_z^2 + \omega_\varphi^2 \pm \sqrt{\left(\omega_z^2 - \omega_\varphi^2\right)^2 + \frac{4\xi}{mI}}\right]$$

Possible parameters to investigate



- Can the transfer of eneregy be incomplete?
- Under what circumstances?
- Why?

- Additions

B. The spiral spring

The spring of the Wilberforce pendulum must be adjusted to the mass and the rotational inertia of the pendulum bob. Assuming an ideal spring, i.e., a negligible mass and rotational inertia (*vide infra*), the angular frequencies of the translational and rotational oscillations are, respectively, $\omega_z = \sqrt{k/m}$ and $\omega_\varphi = \sqrt{\kappa/I}$. The spring and torsion constants are related to material properties via[23]

$$k = \frac{1}{8} \cdot \frac{G}{D^3} \cdot \frac{d^4}{n}, \tag{9}$$

$$\kappa = \frac{1}{3670} \cdot \frac{E}{D} \cdot \frac{d^4}{n}, \tag{10}$$

(G: shear modulus, E: Young modulus, d: wire diameter, D: average coil diameter, n: number of windings), which yields

$$D = \sqrt{\frac{1835}{4} \cdot \frac{\kappa}{k} \cdot \frac{G}{E}}, \tag{11}$$

American Journal of Physics86, 818 (2018); doi: 10.1119/1.5051179